the motion,[11] but, since the free-surface boundary conditions used were still formally linearized, it is uncertain as to whether or not such procedures are a step in the right direction. A completely satisfactory answer to this question would seem to require a wholy numerical solution.

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# Motion of the Center of Gravity of a Variable-Mass Body


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### Nomenclature

| | | |
|---|---|---|
| $B$ | = | body of variable mass moving in space |
| $O'$ | = | a point fixed in space |
| $O$ | = | a point fixed in the body $B$ (origin of body-axes system) |
| $G$ | = | center of gravity of $B$; $G$ changes its position with respect to $O$ as the mass varies |
| $dm$ | = | element of mass of body $B$ located at a point $P$ fixed in the body |
| $\mathbf{R}$ | = | O′P |
| $\mathbf{r}$ | = | OP |
| $\mathbf{R}^{(O)}$ | = | O′O |
| $\mathbf{R}^{(G)}$ | = | O′G |
| $\mathbf{r}^{(G)}$ | = | OG |
| $\mathbf{F}$ | = | external force acting on the body |
| $\mathbf{K}$ | = | reactive force acting on the body, produced by the mass ejection |
| $\omega$ | = | angular velocity of the body $B$ |
| $d/dt$ | = | derivative with respect to a fixed point $O_1$ |
| $\delta/\delta t$ | = | derivative with respect to a point $O$, moving with the body |
| $M$ | = | $\int dm$, the total mass of the body $B$ at the moment under consideration |


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MASS is continuously ejected from some portion on the surface of body $B$. Mass is ejected with a nonzero velocity relative to the point $O$, and consequently the reactive forces are produced. It is assumed that ejected mass, after its separation from the body, does not affect in any way the motion of the body.

Because of the mass ejection, the center of gravity $G$ is displaced relative to the point $O$. The objective of this paper is to derive the equation of motion for the center of gravity $G$.

The equation of motion for the body $B$ can be expressed in the following form:

$$\int (d^2\mathbf{R}/dt^2)dm = \mathbf{F} + \mathbf{K} \tag{1}$$

where the integration is extended over the mass of the body at the time $t$.

For any arbitrary point $P$ of the body $B$, we can write

$$d^2\mathbf{R}/dt^2 = (d^2\mathbf{R}^{(0)}/dt^2) + \omega \times \mathbf{r} + \omega \times (\omega \times \mathbf{r}) \tag{2}$$

or, integrating over the mass of the body $B$, we can write

$$\int (d^2\mathbf{R}/dt^2)dm = \int (d^2\mathbf{R}^{(0)}/dt^2)dm + \int \omega \times \mathbf{r}dm + \int \omega \times (\omega \times \mathbf{r})dm \tag{3}$$

Since

$$\int \mathbf{r}dm = M\mathbf{r}^{(G)} \tag{4}$$

Eq. (3) can be written in the form

$$\int (d^2\mathbf{R}/dt^2)dm = M[(d^2\mathbf{R}^{(0)}/dt^2) + \dot{\omega} \times \mathbf{r}^{(G)} + \omega \times (\omega \times \mathbf{r}^{(G)})] \tag{5}$$

Since $G$ is not fixed in the body $B$,

$$d^2\mathbf{R}^{(G)}/dt^2 = (d^2\mathbf{R}^{(0)}/dt^2) + \omega \times \mathbf{r}^{(G)} + \omega \times (\omega \times \mathbf{r}^{(G)}) + 2\omega \times (\delta\mathbf{r}^{(G)}/\delta t) + (\delta^2\mathbf{r}^{(G)}/\delta t^2) \tag{6}$$

Combining Eqs. (1, 5, and 6), we can write

$$M\frac{d^2\mathbf{R}^{(G)}}{dt^2} = \mathbf{F} + \mathbf{K} + 2M\omega \times \frac{\delta r^{(G)}}{\delta t} + M\frac{\delta^2\mathbf{r}^{(G)}}{\delta t^2} \tag{7}$$

which represents the equation of motion for the variable center of gravity of the variable-mass body.

# Flutter Characteristics of Titanium Alloys


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### Nomenclature

| | | |
|---|---|---|
| $E$ | = | modulus of elasticity |
| $I$ | = | second moment of area |
| $K$ | = | constant |
| $M$ | = | bending moment |
| $V$ | = | velocity |
| $b$ | = | one-half of the chord at reference station |
| $c$ | = | distance from neutral axis to outer fiber of section |
| $w$ | = | weight |
| $\mu$ | = | mass ratio |
| $\sigma$ | = | bending stress |
| $\psi$ | = | flutter parameter ratio |
| $\omega$ | = | circular frequency |

### Introduction

MUCH has been done to perfect materials whose performance characteristics can meet the demands of high-speed flight at temperatures well above the functional range of


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**Table 1 Structural properties**

| Alloy | σ (tensile yield) × 10³ psi[a] Room temp. | 400°F | 600°F | 800°F | 1000°F | E (tensile) × 10⁶ psi[a] Room temp. | 400°F | 600°F | 800°F | 1000°F |
|---|---|---|---|---|---|---|---|---|---|---|
| Al-2024-T86 | 67.0 | 50.3 | ... | ... | ... | 10.5 | 9.5 | ... | ... | ... |
| Ti-8Al-1Mo-1V | 125.0 | 93.8 | 87.5 | 75.0 | 65.0 | 18.5 | 16.8 | 16.3 | 14.6 | 12.2 |
| Ti-6Al-4V | 150.0 | 103.5 | 96.0 | 88.5 | 72.0 | 16.0 | 14.2 | 13.1 | 11.9 | 8.0 |
| Ti-5Al-4FeCr | 150.0 | 115.5 | 103.5 | 91.5 | 72.0 | 16.5 | 14.5 | 13.7 | 12.5 | Not available |

[a] Strengths are for temperature exposure from ½ to 1 hr.

aluminum alloys. Many useful criteria have been established with which to evaluate these alloys from the standpoint of structural strength and fatigue life. Flutter characteristics represent yet another area to be considered. A flutter ratio is developed below for the purpose of comparing the flutter properties of titanium alloys to those of aluminum alloys.

### Flutter Ratio

Given a specific flight vehicle configuration designed in aluminum, it is desired to compare its flutter characteristics to those of the same vehicle designed in titanium. Hence, for the purpose of this development, the external load distribution, the vehicle planform, and the parameter $c$ are assumed to be constant for both configurations. Using these assumptions, the flexural strength equation

$$\sigma = Mc/I \tag{1}$$

yields

$$\sigma_A/\sigma_T = I_T/I_A \tag{2}$$

where the subscripts $A$ and $T$ refer to aluminum and titanium, respectively. The general equation for the fundamental frequency of a beam is

$$\omega = K(EI/w)^{1/2} \tag{3}$$

Placing Eq. (2) into a similar ratio formed from Eq. (3) yields

$$\omega_A{}^2/\omega_T{}^2 = E_A\sigma_T w_T/E_T\sigma_A w_A \tag{4}$$

Although Eq. (4) expresses the ratio of bending frequencies, it should be noted that the torsional frequencies would be in approximately the same ratio. Let

$$\psi = \frac{V_T/b_T\omega_T\mu_T{}^{1/2}}{V_A/b_A\omega_A\mu_A{}^{1/2}} \tag{5}$$

where $V/b\omega\mu^{1/2}$ is the well-known dimensionless flutter parameter. Placing Eq. (4) into Eq. (5) yields the flutter parameter ratio

$$\psi = (E_A\sigma_T/E_T\sigma_A)^{1/2} \tag{6}$$

### Example

Table 1 lists the structural properties of a standard aluminum alloy and those of a number of titanium alloys currently being considered for use in aircraft structures. Figure 1 contains curves of $\psi$ vs temperature for each of the alloys considered. $\psi$ was computed keeping the ratio $E_A/\sigma_A$ as a constant reference value, at room temperature.

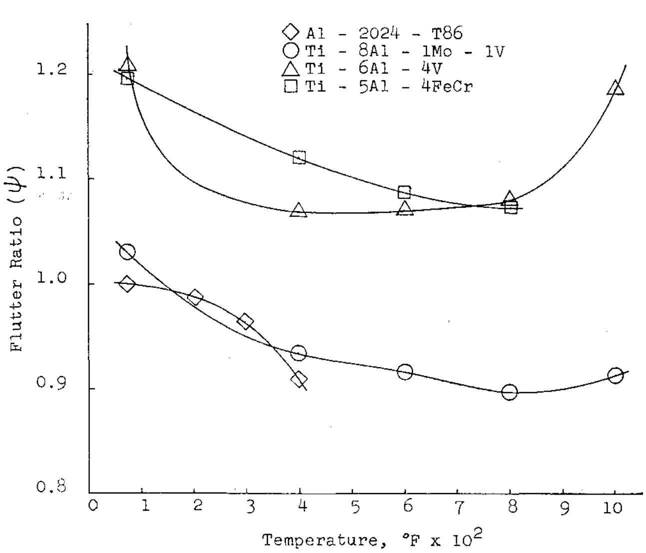


**Fig. 1 Flutter ratio vs temperature for constant reference value of $E_A/\sigma_A$ at room temperature.**

### Concluding Remarks

It should be noted that the development presented herein was based on a predetermined set of conditions. However, as can be seen, the method itself is general. It is recognized that other considerations would dictate changes in the structural design when converting from one material to another. Thus, Fig. 1 illustrates the change in flutter characteristics that result from substituting titanium for aluminum while designing for the same percent of yield stress.

# Equations of Motion for Spin Stabilization Analysis in Terms of Euler Angles

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MOST of the literature on spin stabilization analysis of space vehicles approaches the problem by first solving the Euler equations of rotational motion in terms of body rates (see Ref. 1 for a recent example). This approach has been especially prevalent when the spinning body is influenced by external torques, which are usually expressed in body coordinates. When necessary, the analyses then proceed with coordinate transformations of the body rates into Euler-angle rates so that subsequent integration will result in analytical time histories of the spin-axis Euler angles with respect to inertial space. A classic example of this latter step is presented in Ref. 2.

The purpose of this note is to improve upon the preceding procedure of spin stabilization analysis by deriving linearized equations of satellite rotational motion directly in terms of Euler angles. The equations will include terms containing external torques so that substitution of proper expressions for these torques will allow direct solution for a variety of cases, such as responses to impulsive control torques or natural disturbance torques.

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